# Interaction between particles in a $1/\lambda$-statistics anyon gas.

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(February 1, 2008)

We analyze the dynamics between $1/\lambda$-fractional statistics particles (anyons) in an exact three-body solution of the Sutherland Hamiltonian. We show that anyons interact by means of a short-range attraction. The interaction dictates important features of the spectral function for a charge-1 particle.

PACS numbers: 71.10.-w , 71.10.Pm , 73.20.-r

Particles constrained to move in a plane can exhibit exchange statistics intermediate between Fermi statistics and Bose statistics [1]. Intermediate-statistics particles are called anyons. Anyons occur in nature as the bulk excitations of Fractional Quantum Hall (FQH) samples [2]. In a FQH sample at filling fraction $\nu = 1/m$, the elementary charged excitation is an anyon whose total charge is $e/m$ and has $1/m$ statistics. A similar fractionalization of electric charge and statistics takes place within the edge excitations of such samples. Therefore, edges of a FQH-sample provide a physical realization of a one-dimensional anyon gas [3]. Low-energy, long wavelength Quantum Hall edge excitations are described by a chiral Luttinger liquid model [3]. The Luttinger liquid formalism, however, cannot provide a description of processes with high momentum transfer between anyons (short-distance physics). Short-distance anyon physics is what we study in this paper, in an exact solution of a simple one-dimensional model with $1/\lambda$-statistics excitations ($\lambda$ being a positive integer number): the Sutherland model [4]. The Hamiltonian for $N$ particles lying at $x_1, \ldots, x_N$ on a circle of length $L$ and constant density $\frac{N}{L} = 1$ is:

$$H_S = \left(\frac{2\pi}{N}\right)^2 \left[\sum_{i=1}^{N}\left(z_i \frac{\partial}{\partial z_i}\right)^2 \right.$$

$$\left. - \lambda(\lambda-1)\sum_{i\neq j=1}^{N}\frac{z_i z_j}{(z_i - z_j)^2}\right] \qquad (1)$$

where $z_i = \exp[\frac{2\pi i x_i}{L}]$.

By generalizing the formalism developed in [5], we construct and analyze the Schrödinger equation for any number of anyons. As a result, we show that anyons interact via a short-range attraction. When anyons are far apart, the probability of finding one at a certain position is essentially independent of the positions of the others, showing that anyons are free when far enough away from each other. When any number $M$ ($\leq \lambda$) of anyons come together there is an increase of probability. The maximum enhancement of probability occurs when $\lambda$ of them come together in the $1/\lambda$ statistics anyon gas. The short distance probability enhancement is the progeny of the short distance attraction; anyon attraction largely enhances the probability of configurations with anyons at the same place. Although the contribution to the total energy due to the interaction disappears in the thermodynamic limit, the enhancement does not. Instead, the peak value of the probability increases with $N$, and diverges in the thermodynamic limit. The enhancement is the matrix element for a charge-1 excitation to decay into $\lambda$-anyons. In the thermodynamic limit, the charge-1 excitation completely loses its integrity; as soon as a charge-1 particle/hole is created in the system, it immediately breaks up into $\lambda$ anyons. In particular for the $\lambda = 3$ case we display the relevant physics of anyon attraction in Fig.1 where we plot the square modulus of the three-anyon wavefunction as a function of the separation between two anyons at a time.

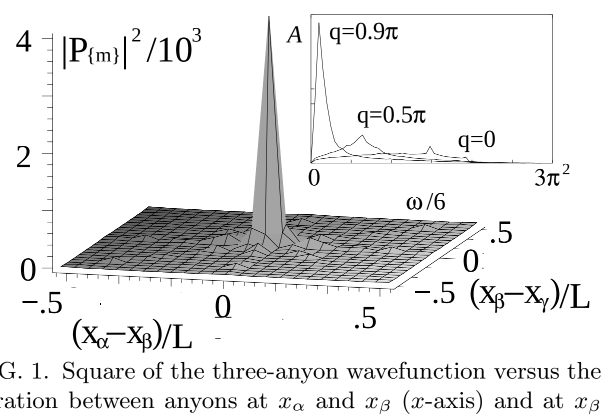


FIG. 1. Square of the three-anyon wavefunction versus the separation between anyons at $x_\alpha$ and $x_\beta$ ($x$-axis) and at $x_\beta$ and $x_\gamma$ ($y$-axis). **Inset**: spectral function vs. $\omega$ at $q = 0.9\pi$, $q = 0.5\pi$, $q = 0$.

Let us start the mathematical derivation of anyon interaction. The ground state of $H_S$ is

$$\Psi_{\rm GS}(z_1,\ldots,z_N)=\prod_{i<j=1}^{N}(z_i-z_j)^{\lambda}\prod_{t=1}^{N}z_t^{-\lambda(N-1)/2} \tag{2}$$

with energy $E_0=\left(\frac{2\pi}{N}\right)^2\lambda^2\frac{N}{12}(N^2-1)$.

Elementary excitations on top of such a ground state are the one-dimensional analogs of Laughlin's anyons for a $\nu=1/\lambda$ FQH-fluid. We create a $1/\lambda$ charge and statistics quasihole at $z_\alpha$ by removing $1/\lambda$ of an electron from the ground state. The excitation is a localized charge defect given by the wavefunction:

$$\Psi_\alpha(z_1,\ldots,z_N)=\prod_{j=1}^{N}(z_j-z_\alpha)\Psi_{\rm GS}(z_1,\ldots,z_N) \tag{3}$$

$\Psi_\alpha$ is not an eigenstate of $H_S$. Energy eigenstates are provided by Fourier transforming $\Psi_\alpha$:

$$\Psi_m(z_1,\ldots,z_N)=\oint\frac{dz_\alpha}{2\pi i(z_\alpha)^{1+m}}\Psi_\alpha(z_1,\ldots,z_N) \tag{4}$$

($m=0,\ldots,N$ and $\oint$ denotes the integral over a closed path surrounding the origin). The corresponding eigenvalues are given by $E_0+E(q_m)=E_0+\left(\frac{2\pi}{N}\right)^2[(N-m)(\lambda m+1)]$. The momentum of $\Psi_m$ is defined as $q_m\equiv\left(\frac{2\pi}{N}\right)[N/2-m]$. In the thermodynamic limit $m/N$ is kept finite, and $E(q_m)=\lambda\left[\pi^2-q_m^2\right]$, where $-\pi\le q_m\le\pi$.

Multi-anyon states are constructed by a procedure analogous to the one used for one-anyon states. The wavefunction for $M$ anyons localized at $z_{\alpha_1},\ldots,z_{\alpha_M}$ is given by:

$$\Psi_{\alpha_1,\ldots,\alpha_M}(z_1,\ldots,z_N)=\prod_{j=1}^{N}\prod_{i=1}^{M}(z_j-z_{\alpha_i})\Psi_{\rm GS}(z_1,\ldots,z_N) \tag{5}$$

A convenient basis for the $M$-anyon eigenstates is given by Fourier transforming Equation (5):

$$\Psi_{m_1,\ldots,m_M}=\oint\Psi_{\alpha_1,\ldots,\alpha_M}\prod_{i=1}^{M}\frac{dz_{\alpha_i}}{2\pi i z_{\alpha_i}^{1+m_i}} \tag{6}$$

with $0\le m_M\le\ldots\le m_1\le N$. These many-anyon plane waves are not energy eigenstates. However, when acting on $\Psi_{m_1,\ldots,m_M}$ with $H_S$, we obtain [4]:

$$[H_S-E_0]\Psi_{\{m\}}=\left(\frac{2\pi}{N}\right)^2[\sum_{i=1}^{M}E(q_{m_i})-\sum_{i<j=1}^{M}(m_i-m_j)]\Psi_{\{m\}}$$

$$+\sum_{i<j=1}^{M}\sum_{k>0}(m_i-m_j+2k)\Psi_{\ldots,m_i+k,\ldots,m_j-k} \tag{7}$$

In Eq.(7) $\{m\}$ is shorthand for $m_1,\ldots,m_M$.

Eq.(7) is a lower-diagonal matrix equation. The energy eigenvalues are the diagonal elements $E_{\{m\}}=E_0+\sum_{i=1}^{M}E(q_{m_i})-\left(\frac{2\pi}{N}\right)^2\sum_{i<j=1}^{M}(m_i-m_j)$. Many-anyon eigenstates of $H_S$, $\Phi_{\{m\}}$, can be constructed from the $\Psi_{\{m\}}$'s by the so-called "squeezing procedure" [4]. The $(m_i-m_j)$ terms make $E_{\{m\}}$ not be equal to the sum of the energies of the single isolated anyons. Therefore, these terms show an interaction for anyons with large relative momenta, i.e at short distances. The anyon interaction is attractive because of the negative sign.

In the thermodynamic limit, the energy eigenvalues $E_{\{m\}}$ becomes equal to

$$E_0+\sum_{i=1}^{M}E(q_{m_i})=E_0+\lambda\sum_{i=1}^{M}(\pi^2-q_{m_i}^2)\quad . \tag{8}$$

We see that the energy for an $M$-anyon eigenstate just reduces to the sum of the energies of the single isolated anyons. The interaction contribution apparently disappears in the thermodynamic limit. However, as it happens with spinons [5], the interaction plays a crucial role in determining some of the relevant physics of the anyon gas, as we are going to discuss next.

The real-space wavefunction for $M$ anyons localized at $z_{\alpha_1},\ldots,z_{\alpha_M}$ can be expanded in terms of the energy eigenstates:

$$\Psi_{\alpha_1,\ldots,\alpha_M}=\sum_{m_1=1}^{N}\sum_{m_2=0}^{m_1}\cdots\sum_{m_M=0}^{m_{M-1}}$$

$$\prod_{i=1}^{M}z_{\alpha_i}^{m_i}\mathcal{P}_{\{m\}}\left(\frac{z_{\alpha_2}}{z_{\alpha_1}},\frac{z_{\alpha_3}}{z_{\alpha_2}},\ldots,\frac{z_{\alpha_M}}{z_{\alpha_{M-1}}}\right)\Phi_{\{m\}} \tag{9}$$

where the function $\mathcal{P}_{\{m\}}\left(\frac{z_{\alpha_2}}{z_{\alpha_1}},\frac{z_{\alpha_3}}{z_{\alpha_2}},\ldots,\frac{z_{\alpha_M}}{z_{\alpha_{M-1}}}\right)$ is a polynomial in each of its variables. By definition, $\prod_{i=1}^{M}z_{\alpha_i}^{m_i}\mathcal{P}_{\{m\}}\left(\frac{z_{\alpha_2}}{z_{\alpha_1}},\frac{z_{\alpha_3}}{z_{\alpha_2}},\ldots,\frac{z_{\alpha_M}}{z_{\alpha_{M-1}}}\right)$ is the coordinate representation of the wavefuction for $M$ anyons in the state of energy $E_{\{m\}}$.

The Schrödinger equation for $\mathcal{P}_{\{m\}}$ is derived from the identity:

$$\langle\Phi_{\{m\}}|(H_S-E_0)|\Psi_{\alpha_1,\ldots,\alpha_M}\rangle=(E_{\{m\}}-E_0)$$

$$\times\langle\Phi_{\{m\}}|\Phi_{\{m\}}\rangle\prod_{i=1}^{M}z_{\alpha_i}^{m_i}\mathcal{P}_{\{m\}}\left(\frac{z_{\alpha_2}}{z_{\alpha_1}},\frac{z_{\alpha_3}}{z_{\alpha_2}},\ldots,\frac{z_{\alpha_M}}{z_{\alpha_{M-1}}}\right) \tag{10}$$

and from the fact that $H_S$ acts on $\Psi_{\alpha_1,\ldots,\alpha_M}$ as:

$$(H_S-E_0)\Psi_{\alpha_1,\ldots,\alpha_M}=\left(\frac{2\pi}{N}\right)^2\Big[(\lambda N-2)\sum_{i=1}^{M}z_{\alpha_i}\frac{\partial}{\partial z_{\alpha_i}}$$

$$-\lambda \sum_{i=1}^{M} \left( z_{\alpha_i} \frac{\partial}{\partial z_{\alpha_i}} \right)^2 + M^2 N$$

$$- \frac{1}{2} \sum_{i \neq j=1}^{M} \left( \frac{z_{\alpha_i} + z_{\alpha_j}}{z_{\alpha_i} - z_{\alpha_j}} \right) \left( z_{\alpha_i} \frac{\partial}{\partial z_{\alpha_i}} - z_{\alpha_j} \frac{\partial}{\partial z_{\alpha_j}} \right) \Bigg] \Psi_{\alpha_1,\dots,\alpha_M} \tag{11}$$

By putting together Eqs.(9,10,11), we obtain the equation of motion for the relative coordinate M-anyon wavefunction $\mathcal{P}_{\{m\}} \left( \frac{z_{\alpha_2}}{z_{\alpha_1}}, \frac{z_{\alpha_3}}{z_{\alpha_2}}, \dots, \frac{z_{\alpha_M}}{z_{\alpha_{M-1}}} \right)$:

$$\Bigg\{ 2\lambda \sum_{i=1}^{M} m_i z_{\alpha_i} \frac{\partial}{\partial z_{\alpha_i}} + \lambda \sum_{i=1}^{M} \left( z_{\alpha_i} \frac{\partial}{\partial z_{\alpha_i}} \right)^2$$

$$+ \sum_{i<j=1}^{M} \left( \frac{z_{\alpha_i} + z_{\alpha_j}}{z_{\alpha_i} - z_{\alpha_j}} \right) \left( z_{\alpha_i} \frac{\partial}{\partial z_{\alpha_i}} - z_{\alpha_j} \frac{\partial}{\partial z_{\alpha_j}} \right)$$

$$+ \sum_{i<j=1}^{M} (m_i - m_j) \left( \frac{z_{\alpha_i} + z_{\alpha_j}}{z_{\alpha_i} - z_{\alpha_j}} \right)$$

$$+ \sum_{i<j=1}^{M} (m_i - m_j) \Bigg\} \mathcal{P}_{\{m\}} \left( \frac{z_{\alpha_2}}{z_{\alpha_1}}, \frac{z_{\alpha_3}}{z_{\alpha_2}}, \dots, \frac{z_{\alpha_M}}{z_{\alpha_{M-1}}} \right) = 0 \quad . \tag{12}$$

Eq.(12) for three anyons and any $\lambda$ can be solved exactly [6]. In order to illustrate the features of anyon interaction, in the rest of this letter we shall analyze the full solution of Eq.(12) for the case of three anyons and $\lambda = 3$ (the case of two spinons, i.e. two anyons with $\lambda = 2$ has been solved in [5].) $|\mathcal{P}_{\{m\}}(z,w)|^2$, with $z = z_{\alpha_2}/z_{\alpha_1}$ $w = z_{\alpha_3}/z_{\alpha_2}$, is the function we plot in Fig.1, for $\lambda = 3, N = 20, m_1 - m_2 = 10, m_2 - m_3 = 7$.

We can extract physical consequences of anyon attraction by computing the spectral density for the charge-1 excitation. The state where one more charge-1 hole has been created at $\xi$, on top of the $N$-particle ground state, is described by the wavefunction $\Psi_\xi(z_1,\dots,z_N) = \prod_{j=1}^{N}(z_j - \xi)^3 \Psi_{\rm GS}(z_1,\dots,z_N)$. From the definition of $\mathcal{P}_{\{m\}}$ we find the following decomposition for $\Psi_\xi$ in terms of the states $\Phi_{\{m\}}$:

$$\Psi_\xi = \sum_{m_\alpha=0}^{N} \sum_{m_\beta=0}^{m_\alpha} \sum_{m_\gamma=0}^{m_\beta} \xi^{m_\alpha+m_\beta+m_\gamma} \mathcal{P}_{\{m\}}(1,1) \Phi_{\{m\}} \tag{13}$$

Since only positive-energy states propagate, the Green function for a charge-1 hole is defined as $G(\xi,t) = -i\theta(t)\langle \Psi_\xi(t) | \Psi_{\xi=1}(t=0)\rangle$. The Green function corresponds to a physical hole, which is obtained by pulling a real electron out of the system. The spectral density of states, $\mathcal{A}(\omega,q)$ is the imaginary part of the Fourier transform of $G$ with respect to both $t$ and $\xi$, that is:

$$\mathcal{A}(\omega,q) = \sum_{0 \le m_\gamma \le m_\beta \le m_\alpha \le N} \frac{|\mathcal{P}_{\{m\}}(1,1)|^2 \langle \Phi_{\{m\}} | \Phi_{\{m\}} \rangle}{\langle \Psi_{\rm GS} | \Psi_{\rm GS} \rangle}$$

$$\times \, \delta_{m_\alpha+m_\beta+m_\gamma-n} \delta(\omega - (E_{\{m\}} - E_0)) \tag{14}$$

($q = 2\pi n/L$, $0 \le n \le 3N$ is the total momentum of the hole).

In order to compute $\mathcal{A}(\omega,q)$, we need the norm of the energy eigenstates and the enhancements $\mathcal{P}_{\{m\}}(1,1)$. The norms are obtained by following the methods developed in [5] and can be expressed in terms of $\langle \Psi_{\rm GS} | \Psi_{\rm GS}\rangle = \Gamma[3N+1]/\Gamma[3]^N$ [7]. The solution is given by:

$$\frac{\langle \Phi_{\{m\}} | \Phi_{\{m\}} \rangle}{\langle \Psi_{\rm GS} | \Psi_{\rm GS} \rangle} = \prod_{i=1}^{3} \frac{\Gamma[N+1+(i-1)/3]}{\Gamma[N+i/3]} \times$$

$$\prod_{i=1}^{3} \frac{\Gamma[N-m_i+i/3]\Gamma[m_i+(4-i)/3]}{\Gamma[N-m_i+1+(i-1)/3]\Gamma[m_i+1+(3-i)/3]} \times$$

$$\prod_{i<j=1}^{3} \frac{\Gamma[m_i-m_j+(j-i)/3]\Gamma[m_i-m_j+(j-i)/3+1]}{\Gamma[m_i-m_j+1+(j-i-1)/3]\Gamma[m_i-m_j+(j-i+1)/3]} \ .$$

$\mathcal{P}_{\{m\}}(1,1)$ can be computed from equation (12). The derivation will be described in a forthcoming paper [6]. Here, let us just quote the main result:

$$\mathcal{P}_{\{m\}}(1,1) = \frac{\Gamma^2[\frac{1}{3}]}{\Gamma[\frac{2}{3}]} \prod_{i<j=1}^{3} \frac{\Gamma[m_i - m_j + \frac{j-i+1}{3}]}{\Gamma[m_i - m_j + \frac{j-i}{3}]}$$

By inserting in Eq.(14) the formulas for the norms and for the enhancements, it is straightforward to work out $\mathcal{A}(\omega,q)$ in the thermodynamic limit. In this limit, we can trade the sums for integrals over the Brillouin zone of the three anyons, so that Eq.(14) takes the form:

$$\mathcal{A}(\omega,q) = \frac{\Gamma[\frac{1}{3}]}{\Gamma^2[\frac{2}{3}]} \int_{-\pi}^{\pi} \prod_{\rho=\alpha,\beta\gamma} \left( \frac{dq_\rho}{6} \right) \delta[\omega - E(q_\alpha, q_\beta, q_\gamma)]$$

$$\times \left| \frac{(q_\alpha - q_\beta)(q_\beta - q_\gamma)(q_\alpha - q_\gamma)}{(\pi^2 - q_\alpha^2)(\pi^2 - q_\beta^2)(\pi^2 - q_\gamma^2)} \right|^{\frac{2}{3}} \delta[q - q_\alpha - q_\beta - q_\gamma]$$

In the inset of Fig.1, we plot $\mathcal{A}(q,\omega)$ at different values of $q$. We recognize a fundamental feature of the anyon gas in that no definite Landau quasiparticle peak appears. This corresponds to the lack of integrity of the charge-1

hole we discussed before. More importantly, we can distinguish a well-defined sharp peak in the spectral function, whose height crucially depends on $q$. Such a "spectral enhancement" is the enhancement in the matrix element for a hole to decay in three anyons, $\mathcal{P}_{\{m\}}(1,1)$. The dependence on $q$ reflects the dependence of $\mathcal{P}_{\{m\}}(1,1)$ on the relative anyon momenta, the larger the relative momenta, the larger the enhancement. Therefore, the maximum enhancement should appear for values of $q$ that make the differences among anyon momenta as large as possible, namely, when two anyons lie at opposite sites of the Brillouin zone ($q \approx \pm\pi$), while a third one has momentum $q \approx 0$. On the other hand, the double constraint of momentum and energy conservation implies that the number of three-anyon states in which the hole decays is proportional to the size of the intersection between the plane $\sum_i q_i - q = 0$, the sphere $\sum_i q_i^2 = 3\pi^2 - \omega/6$ and the Brillouin zone $[-\pi, \pi]^3$. Such a size is maximum when all three anyons lie at the corners of the Brillouin zone, that is, $q_i \approx \pm\pi$ $i = 1, 2, 3$. The two competing maximum conditions discussed above imply that the maximum in the spectral function in this system does not appear exactly at the emission threshold, as it happens with spinons [5], but at some intermediate value of $\omega$, depending on $q$.

Such a trend clearly appears from the plots in the inset of Fig.1 as a signature of the nonelementarity of the charge-1 excitation and of the corresponding attraction among fractionally charged quasiparticles.

In the present work, the results of [5] have been generalized to excitations of arbitrary statistics $1/\lambda$. The main result is that fractionalized particles of any statistics always interact through a short distance attraction discovered in [5]. We conclude that short distance attraction is a generic feature of the dynamics of fractionalized excitations. Such an attraction has profound consequences in the spectral function of the charge-1 hole. The peculiar distribution of spectral weight should provide a possible experimental evidence of anyon attraction.

In conclusion, it clearly appears that, because of its generality, our result should be a common feature of any anyon gas and, in particular, that it might provide a simple explanation of some of the interesting and puzzling tunelling experimental findings on FQH samples [8,9].

As a mathematical appendix we provide the explicit recursion relation for the coefficients of $\mathcal{P}_{\{m\}}$. We assume a polynomial solution of the form

$$\mathcal{P}_{\{m\}}(z,w) = \sum_{p,q \geq 0} b_{p,q} z^p w^q \ ; \ \ \mathcal{P}_{\{m\}}(0,0) = 1$$

where the summations over $p$ and $q$ stop at some finite indices. The boundary condition implies $b_{0,0} = 1$. Moreover, by inserting into Eq.(12) the polynomial expansion, it is straightforward to work out a recursion relation for the coefficients $b_{p,q}$:

$$\begin{aligned}
&\left[(\mu-p+q)(p+2)+(\nu-q-2)(q+2)+\frac{p+q+4}{3}\right]b_{p+2,q+2} = \\
&\left[(\mu-p-1+q)(p+2)+(\nu-q-1)(q+1)-1+\frac{\nu+q-2p}{3}\right]b_{p+2,q+1} \\
&+\left[(\mu-p+q+1)(p+1)+(\nu-q-2)(q+2)-1+\frac{\mu+p-2q}{3}\right]b_{p+1,q+2} \\
&-\left[(\mu-p+q-1)(p+1)+(\nu-q)q+\frac{\mu+2\nu+1+p-2q}{3}\right]b_{p+1,q} \\
&-\left[(\mu-p+q+1)p+(\nu-q-1)(q+1)+\frac{\nu+2\mu+1+q-2p}{3}\right]b_{p,q+1} \\
&+\left[\mu p+\nu q+pq-p^2-q^2+\frac{2(\mu+\nu)-p-q}{3}\right]b_{p,q}
\end{aligned} \tag{15}$$

where, if either $p$ or $q$ (or both of them) is $< 0$, $b_{p,q} = 0$, $b_{\mu+k,j} = 0$ for $k > j$, $b_{\mu+\nu+i,q} = 0$ for $i > 0$.

This work has been conceived at a suggestion by R. B. Laughlin. His support and his advice to us in writing this paper have been extremely precious. We kindly thank him. We acknowledge interesting discussions with A. Tagliacozzo, P. Sodano, G. Maiella and M. Bergére.

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